$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ $E_b = \sigma T^4$

ATKINSON SCIENCE

$e^{i\pi} = -1$ $\frac{u}{u_\tau} = \frac{1}{\kappa}\ln\frac{yu_\tau}{\nu} + C$

THEORY GUIDE

Admittance Method

1 One−Dimensional Transient Heat Conduction

Keith Atkinson

23 October 2020

Atkinson Science welcomes your comments on this Theory Guide. Please send an email to keith.atkinson@atkinsonscience.co.uk.

Contents

1 Introduction 5
2 Differential equation of one–dimensional heat conduction 7
3 Properties of the differential equation 11
3.1 Definition of a linear partial differential equation 11
3.2 Boundary–value problems 11
3.3 Superposition principle 12
4 Dimensionless form 13
5 Example 1 15
6 Example 2 27
7 Example 3 33
8 References 37
9 Appendix 39

Figures

Figure 1 Flow of heat into a planar building structure 7
Figure 2 Flow of heat through a thin planar layer 8
Figure 3 $\theta(X,T)$ vs. X and ΩT for oscillations on the outer surface of the steel plate 20
Figure 4 $\theta(X,T)$ vs. X and ΩT for oscillations on the outer surface of the timber board 25
Figure 5 $\theta(X,T)$ vs. X and ΩT for oscillations on the inner surface of the timber board 29
Figure 6 $\theta(X, T)$ vs. X and ΩT for oscillations on both sides of the timber board 31
Figure 7 Solutions to the third and fourth boundary-value problems 34
Figure 8 $\theta(X, T)$ vs. X and ΩT with straight-line mean value 35

Tables

Table 1 $\theta(X,T)$ vs. X and ΩT for oscillations on the outer surface of the steel plate 19
Table 2 $\theta(X,T)$ vs. X and ΩT for oscillations on the outer surface of the timber board 24
Table 3 $\theta(X,T)$ vs. X and ΩT for oscillations on the inner surface of the timber board 28
Table 4 $\theta(X,T)$ vs. X and ΩT for oscillations on both sides of the timber board 30

1 Introduction

All building materials have *thermal mass*, which is the ability to store heat. In general, dense materials have a higher thermal mass than light materials. Concrete has a large thermal mass, so on a hot summer's day it can absorb a large quantity of heat from the environment without undergoing a large temperature change. Then in the evening it can release the heat back to the environment. In this way, the concrete acts to shield the building from large fluctuations in temperature.

Concrete inside a building can reduce night-time cooling by releasing heat absorbed during the day. However, the concrete can only be effective if its surface is exposed to the internal environment. An air-cavity and an outer layer of plasterboard will act to insulate the concrete and reduce its effectiveness.

Designers of buildings often make use of thermal mass as a way of regulating the internal temperature of a building. Given a fluctuation in temperature on one or both sides of a building structure, it is possible to determine the heat fluxes from the two sides that will result. One of the most popular methods of determining the heat fluxes is the *admittance method*. The admittance method is strictly only valid when the heat flow is one-dimensional. In construction, this requirement is not particularly limiting. Most of the structures that make up a building are large planar structures and the direction of heat flow is only significant in the direction normal to the outer surfaces.

In this theory guide we shall derive the equation of one-dimensional transient heat conduction through a composite wall. Then we will show how the time-varying temperature can be calculated for the case of a slab with uniform properties when a sinusoidal temperature variation is applied to one side and the temperature of the other side is held constant.

The mathematical analysis becomes much more complicated when it is applied to a composite slab. However, designers involved in calculations of thermal mass are usually more interested in the heat fluxes into and out of a structure than in the time-varying temperature profile through it. The admittance method is particularly appealing because it calculates the heat fluxes but avoids the task of calculating the temperature profile. The second, third and fourth reports in this series, Refs. [1] to [3], set out the mathematical basis of the admittance method. The fifth report, Ref. [4], shows how the admittance method can be used to calculate three important dynamic thermal parameters of a planar composite structure: the decrement factor and its time lag, the thermal admittance and its time lead, and the surface factor and its time lag. For many common composite structures these parameters have been tabulated. Ref. [5] gives tables of parameter values for different types of walls, roofs, etc. All of the theory guides contain numerous worked examples which highlight the key principles in the text and illustrate how the method is used.

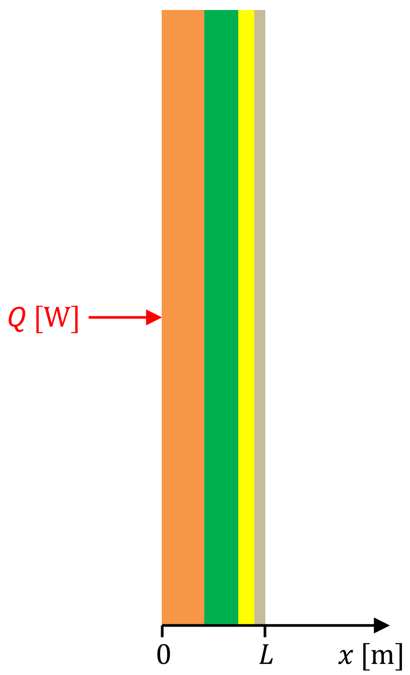
2 Differential equation of one−dimensional heat conduction

In this theory guide we shall derive the differential equation of transient one-dimensional heat conduction. We shall then solve the equation in order to determine the temperature in a flat plate when a sinusoidal oscillation in temperature is applied to one side of the plate and the temperature on the other side is held constant.

The mathematics includes partial differential equations (PDEs), the solution of boundary-value problems involving PDEs, and algebra involving complex numbers. All of these subjects are covered in most textbooks on advanced mathematics for engineers and scientists. Ref. [6] covers all of the mathematics in this report and others in this series on the admittance method.

The envelope of a building typically consists of planar structures whose height and width are much greater than their thickness. The structure may be built up from several layers of different materials as shown in Figure 1. To a close approximation, we may assume that the flow of heat is only significant in the direction x normal to the two faces of the structure. In this report and succeeding reports in this series we shall take x to be zero on the outdoor surface of the wall and x to be equal to L on the indoor surface. A heat flow Q will be taken to be positive if it is in the positive x direction.

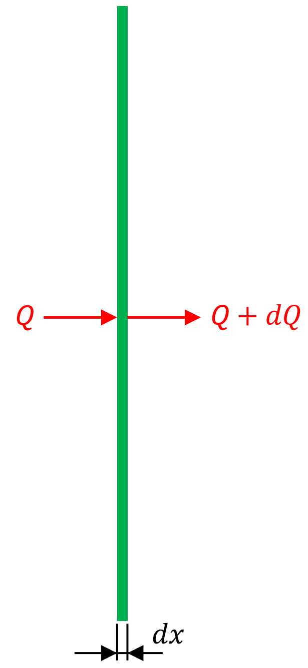
Figure 1 Flow of heat into a planar building structure



Assuming that all the layers are solids, then the flow of heat will be by thermal conduction. Each layer has its own thermal properties: density ρ, specific heat capacity C, and thermal conductivity k. If there is an air layer there may also be some heat flow by convection, but we can usually specify an equivalent thermal conductivity which accounts for the convection. Since the thermal properties vary from layer to layer, we must regard them as functions of x: $\rho = \rho(x)$, $C = C(x)$, $k = k(x)$.

To derive the differential equation of heat conduction through the structure, we must consider a thin planar layer with infinitesimal thickness dx [m] and thermal properties $\rho(x)$ [kg m^{-3}], $C(x)$ [J kg^{-1} K^{-1}] and $k(x)$ [W m^{-1} K^{-1}], as shown in Figure 2.

Figure 2 Flow of heat through a thin planar layer



The heat flow Q [W] into the left–hand face of the layer is given by Fourier's law:

$$Q = -A\,k(x)\frac{\partial\theta}{\partial x}$$

where A [m^2] is the facial area of the layer. The heat flow out of the right–hand face is given by

$$Q + dQ = Q + \frac{\partial Q}{\partial x}dx = -A\,k(x)\frac{\partial\theta}{\partial x} - A\frac{\partial}{\partial x}\left[k(x)\frac{\partial\theta}{\partial x}\right]dx$$

The net rate of heat inflow is therefore

$$Q - (Q + dQ) = -A\,k(x)\frac{\partial\theta}{\partial x} + A\,k(x)\frac{\partial\theta}{\partial x} + A\frac{\partial}{\partial x}\left[k(x)\frac{\partial\theta}{\partial x}\right]dx$$

$$= \frac{\partial}{\partial x}\left[k(x)\frac{\partial\theta}{\partial x}\right]A\,dx \qquad (2.1)$$

Heat may be stored in the layer or released in the layer. If the temperature of the layer rises at the rate $\partial\theta/\partial t$ then the rate of energy storage is

$$dQ = mC(x)\frac{\partial\theta}{\partial t} \qquad (2.2)$$

where m [kg] is the mass of the layer, which is given by

$$m = \rho(x)A\,dx \qquad (2.3)$$

Substituting (2.3) into (2.2) gives

$$dQ = \rho(x)C(x)\frac{\partial\theta}{\partial t}A\,dx \qquad (2.4)$$

The net rate of heat inflow must be equal to the rate of heat storage, so we can equate (2.1) and (2.4):

$$\rho(x)C(x)\frac{\partial\theta}{\partial t}A\,dx = \frac{\partial}{\partial x}\left[k(x)\frac{\partial\theta}{\partial x}\right]A\,dx$$

or

$$\rho(x)C(x)\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left[k(x)\frac{\partial\theta}{\partial x}\right] \qquad (2.5)$$

This is the one−dimensional equation of transient heat conduction for a material whose thermal properties vary in the heat flow direction. If the thermal properties are constant, then (2.5) simplifies to

$$\frac{\partial\theta}{\partial t} = \alpha\frac{\partial^2\theta}{\partial x^2} \qquad (2.6)$$

where $\alpha = k/(\rho C)$ [m^2 s^{-1}] is the (constant) thermal diffusivity of the material.

3 Properties of the differential equation

3.1 Definition of a linear partial differential equation

The general *linear partial differential equation* of order two in two independent variables has the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \qquad (3.1)$$

where A, B, …, G may depend on x and y but not on u. A second order equation with independent variables x and y which does not have the form (3.1) is called *nonlinear*. If $G = 0$ the equation is called *homogeneous*, while if $G \neq 0$ the equation is called *non−homogeneous*.

From the foregoing definitions it follows that the one−dimensional equation of transient heat conduction for a uniform solid (2.6) is a homogeneous linear partial differential equation of order two.

3.2 Boundary−value problems

In a *boundary−value problem* involving a partial differential equation we try to find a solution of the equation subject to *boundary conditions*.

Suppose the temperature is made to oscillate on the side $x = 0$ of a solid wall with uniform properties so that $\theta(0,\, t) = A\,\cos\omega t$ and the temperature is held constant at zero on the side $x = L$ so that $\theta(L, t) = 0$. At $t = 0$, the temperature is zero so $\theta(x,\, 0) = 0$. The boundary−value problem is to find the temperature $\theta(x,\, t)$ in the region $0 < x < L$, $t > 0$ that satisfies the partial differential equation

$$\frac{\partial \theta}{\partial t} = \alpha\frac{\partial^2 \theta}{\partial x^2} \qquad 0 < x < L, t > 0 \qquad (3.2)$$

subject to the boundary conditions

$$\theta(0, t) = A\cos\omega t \qquad t \geq 0 \qquad (3.3)$$

$$\theta(L, t) = 0 \qquad t \geq 0 \qquad (3.4)$$

$$\theta(x, 0) = 0 \qquad 0 < x < L \qquad (3.5)$$

3.3 Superposition principle

This principle states that if u_1, u_2, ... , u_n are solutions of a homogeneous linear partial differential equation, then $c_1 u_1 + c_2 u_2 + \ldots + c_n u_n$ where c_1, c_2, …, c_n are constants is also a solution.

Suppose the temperature is made to oscillate on both sides of a solid wall with uniform properties so that $\theta(0, t) = A \cos\omega t$ and $\theta(L, t) = B \cos(\omega t + \phi)$. At $t = 0$, the temperature is zero as before so $\theta(x, 0) = 0$. To find the solution of this boundary−value problem we can invoke the superposition principle and add the solution of the boundary−value problem (3.2), …, (3.5) to the solution of the boundary−value problem:

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} \quad 0 < x < L, t > 0 \quad (3.6)$$

$$\theta(0, t) = 0 \quad t \geq 0 \quad (3.7)$$

$$\theta(L, t) = B \cos(\omega t + \phi) \quad t \geq 0 \quad (3.8)$$

$$\theta(x, 0) = 0 \quad 0 < x < L \quad (3.9)$$

Now suppose on the side $x = 0$ the temperature oscillates about $\theta = \theta_0$ so that $\theta(0, t) = \theta_0 + A \cos\omega t$ and on the side $x = L$ the temperature oscillates about $\theta = \theta_L$ so that $\theta(L, t) = \theta_L + B \cos(\omega t + \phi)$. We can solve this boundary value problem by adding the solutions (3.2), …, (3.5) and (3.6), …, (3.9) to the solution of the boundary−value problem:

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} \quad 0 < x < L, t > 0 \quad (3.10)$$

$$\theta(0, t) = \theta_0 \quad t \geq 0 \quad (3.11)$$

$$\theta(L, t) = \theta_L \quad t \geq 0 \quad (3.12)$$

$$\theta(x, 0) = 0 \quad 0 < x < L \quad (3.13)$$

4 Dimensionless form

Mathematicians have discovered analytical solutions to a number of boundary–values problems involving the constant–property equation (2.6). When a boundary–value problem involves the variable–property equation (2.5), then very often there is no analytical solution and mathematicians have to resort to a numerical method. Whether an analytical method or a numerical method is employed, the process of solving for the time–varying temperature profile involves a considerable number of individual calculations.

If we put (2.6) into dimensionless form before solving a particular boundary–value problem, then the solution generated can be applied to the same boundary–value problem, but with different dimensional values of length, time, and temperature, provided the dimensionless values of the independent variables are the same. Consequently, when an analytical solution is found, wherever possible, it is based on the dimensionless equation and expressed in dimensionless form.

Recalling the constant–property equation (2.6),

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} \qquad (2.6)$$

This gives the temperature θ at a distance x after a time t in a slab with thermal diffusivity α. Suppose we let L represent the thickness of the slab and θ_o a particular temperature, say the maximum or minimum temperature at time zero. Then let t_o be a timescale that we shall define later. We can form the dimensionless variables X, T and Θ:

$$X = \frac{x}{L}$$

$$T = \frac{t}{t_o}$$

$$\Theta = \frac{\theta}{\theta_o}$$

In this report we shall always take $\theta_o = 1$°C. The derivative $\partial\theta/\partial t$ on the left–hand side of (2.6) can be expressed in terms of T and Θ as follows.

$$\frac{\partial \theta}{\partial t} = \frac{dT}{dt}\frac{\partial(\Theta\theta_o)}{\partial T} = \frac{\theta_o}{t_o}\frac{\partial \Theta}{\partial T} \qquad (4.1)$$

The derivative $\partial^2\theta/\partial x^2$ on the right–hand side of (2.6) can be expressed in terms of X and Θ as follows. First, take

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial X}\frac{\partial X}{\partial x} = \frac{\partial \theta}{\partial X}\frac{1}{L}$$

then take

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right) = \frac{dX}{dx}\frac{\partial}{\partial X}\left(\frac{1}{L}\frac{\partial \theta}{\partial X}\right) = \frac{1}{L^2}\frac{\partial^2 \theta}{\partial X^2} = \frac{\theta_o}{L^2}\frac{\partial^2 \Theta}{\partial X^2} \qquad (4.2)$$

Substituting (4.1) and (4.2) into (2.6) gives

$$\frac{\theta_o}{t_o}\frac{\partial\Theta}{\partial T} = \alpha\frac{\theta_o}{L^2}\frac{\partial^2\Theta}{\partial X^2}$$

or

$$\frac{\partial\Theta}{\partial T} = \alpha\frac{t_o}{L^2}\frac{\partial^2\Theta}{\partial X^2}$$

If we define t_o as L^2/α then we have the dimensionless form of (2.6):

$$\frac{\partial\Theta}{\partial T} = \frac{\partial^2\Theta}{\partial X^2} \qquad (4.3)$$

Solving this equation yields $\Theta(X, T)$.

Suppose a slab has thermal diffusivity α and thickness L and one side is subjected to a sinusoidal temperature oscillation about zero with amplitude θ_o. The other side of the slab is kept at zero temperature. To obtain the solution $\theta(x, t)$ from $\Theta(X, T)$, we convert the dimensionless independent variables X and T to the dimensional independent variables x and t:

$$x = XL$$

$$t = Tt_o = T\frac{L^2}{\alpha}$$

Then for each Θ at X and T, we have $\theta = \Theta\theta_o$ at x and t.

5 Example 1

A steel plate has a thickness L of 20 mm. On the outer surface of the plate, $x = 0$, the temperature is made to oscillate sinusoidally about 0°C with a period of 10 minutes and an amplitude of 50°C. On the inner surface, $x = L$, the temperature is held constant at 0°C. The plate is initially at 0°C.

(a) Calculate the temperature in the plate once sufficient time has elapsed for the temperature to become periodic. The plate has the thermal properties $\rho = 7800$ kg m^{-3}, $C = 480$ J kg^{-1} K^{-1}, $k = 45$ W m^{-1} K^{-1}.

(b) Repeat the calculation for a timber board of the same thickness. The board has the thermal properties $\rho = 480$ kg m^{-3}, $C = 1680$ J kg^{-1} K^{-1}, $k = 0.12$ W m^{-1} K^{-1}.

(a) To solve this problem we can make use of the following boundary–value problem in dimensionless form:

$$\frac{\partial \theta}{\partial T} = \frac{\partial^2 \theta}{\partial X^2} \qquad 0 < X < 1, T > 0$$

$$\theta(0, T) = A_0 \cos \Omega T \qquad t \geq 0$$

$$\theta(1, T) = 0 \qquad T \geq 0$$

$$\theta(X, 0) = 0 \qquad 0 < X < 1$$

The dimensionless temperature θ in the slab is initially zero everywhere. After a period of time has elapsed the initial state of the slab will have negligible influence on the temperature in the slab and the temperature will become periodic. When this is the case we would expect $\theta(X, T)$ to have the form:

$$\theta(X, T) = A(X) \cos[\Omega T + \phi(X)] \qquad (5.1)$$

We can write this equation as

$$\theta(X, T) = \mathrm{Re}\{A(X) e^{j\phi(X)} e^{i\Omega T}\} = \mathrm{Re}\{U(X) e^{j\Omega T}\}$$

where

$$U(X) = A(X) e^{j\phi(X)}$$

and $j = \sqrt{-1}$. Ref. [7] gives the following equation for the complex function $U(X)$:

$$U(X) = A_0 \frac{\exp\left[\sqrt{\frac{\Omega}{2}}(1+j)(1-X)\right] - \exp\left[-\sqrt{\frac{\Omega}{2}}(1+j)(1-X)\right]}{\exp\left[\sqrt{\frac{\Omega}{2}}(1+j)\right] - \exp\left[-\sqrt{\frac{\Omega}{2}}(1+j)\right]} \qquad (5.2)$$

The amplitude $A(X)$ of $\Theta(X,T)$ is then

$$A(X) = \sqrt{\{\mathrm{Re}[U(X)]\}^2 + \{\mathrm{Im}[U(X)]\}^2}$$

and the phase $\phi(X)$ of $\Theta(X,T)$ is

$$\phi(X) = \arctan\left\{\frac{\mathrm{Im}[U(X)]}{\mathrm{Re}[U(X)]}\right\}$$

In the Appendix we show how to separate the real part of $U(X)$ from the imaginary part so that $A(X)$ and $\phi(X)$ can be calculated and we can plot $\Theta(X,T)$ using (5.1).

The period of the oscillations p is 10 minutes (600 s). The period is a timescale, so it must be made dimensionless as follows:

$$P = \frac{p}{t_o} = \frac{p\alpha}{L^2}$$

The angular frequency term Ω in (5.1) and (5.2) is then

$$\Omega = \frac{2\pi}{P} = \frac{2\pi L^2}{p\alpha}$$

We can use the solution of (5.1) to solve any boundary−value problem like the present one, provided Ω is the same. We can consider any amplitude because the solution is linear in A_0.

The thermal diffusivity α of the steel plate is

$$\alpha = \frac{k}{\rho C} = \frac{45}{7800 \times 480} = 12.02 \times 10^{-6}\ \mathrm{m^2\ s^{-1}}$$

so the angular frequency term Ω is

$$\Omega = \frac{2\pi \times 0.02^2}{600 \times 12.02 \times 10^{-6}} = 0.3485\ \mathrm{rad}$$

From the Appendix

$$U(X) = A_0 \frac{(e^a - e^{-a})\cos a\left[e^{a(1-X)} - e^{-a(1-X)}\right]\cos a(1-X) + (e^a + e^{-a})\sin a\left[e^{a(1-X)} + e^{-a(1-X)}\right]\sin a(1-X)}{(e^a - e^{-a})^2\cos^2 a + (e^a + e^{-a})^2\sin^2 a}$$

$$+jA_0 \frac{(e^a - e^{-a})\cos a\left[e^{a(1-X)} + e^{-a(1-X)}\right]\sin a(1-X) - (e^a + e^{-a})\sin a\left[e^{a(1-X)} - e^{-a(1-X)}\right]\cos a(1-X)}{(e^a - e^{-a})^2\cos^2 a + (e^a + e^{-a})^2\sin^2 a} \quad (\mathrm{A.1})$$

where

$$a = \sqrt{\frac{\Omega}{2}} = \sqrt{\frac{0.3485}{2}} = 0.4174$$

Substituting the value of a gives

$$(e^a - e^{-a})\cos a = (e^{0.4174} - e^{-0.4174})\cos 0.4174 = 0.7854820$$

$$(e^a + e^{-a})\sin a = (e^{0.4174} + e^{-0.4174})\sin 0.4174 = 0.8824288$$

$$(e^a - e^{-a})^2\cos^2 a + (e^a + e^{-a})^2\sin^2 a = 0.7854820^2 + 0.8824288^2 = 1.3956626$$

$$\frac{(e^a - e^{-a})\cos a}{(e^a - e^{-a})^2\cos^2 a + (e^a + e^{-a})^2\sin^2 a} = \frac{0.7854820}{1.3956626} = 0.5628022$$

$$\frac{(e^a + e^{-a})\sin a}{(e^a - e^{-a})^2\cos^2 a + (e^a + e^{-a})^2\sin^2 a} = \frac{0.8824288}{1.3956626} = 0.6322651$$

Substituting into (A.1) gives

$$U(X) = A_0\left\{0.5628022\left[e^{a(1-X)} - e^{-a(1-X)}\right]\cos a(1-X) + 0.6322651\left[e^{a(1-X)} + e^{-a(1-X)}\right]\sin a(1-X)\right\}$$

$$+jA_0\left\{0.5628022\left[e^{a(1-X)} + e^{-a(1-X)}\right]\sin a(1-X) - 0.6322651\left[e^{a(1-X)} - e^{-a(1-X)}\right]\cos a(1-X)\right\}$$

where $A_0 = 50°\mathrm{C}$ and $a = 0.4174$.

In Table 1, column 1 shows X from 0 to 1 in intervals of 0.02. Columns 2 and 3 show the real and imaginary parts of $U(X)$, column 4 shows the amplitude $A(X)$ of $\Theta(X,T)$:

$$A(X) = \sqrt{\{\mathrm{Re}[U(X)]\}^2 + \{\mathrm{Im}[U(X)]\}^2}$$

and column 5 shows the phase $\phi(X)$ of $\Theta(X,T)$:

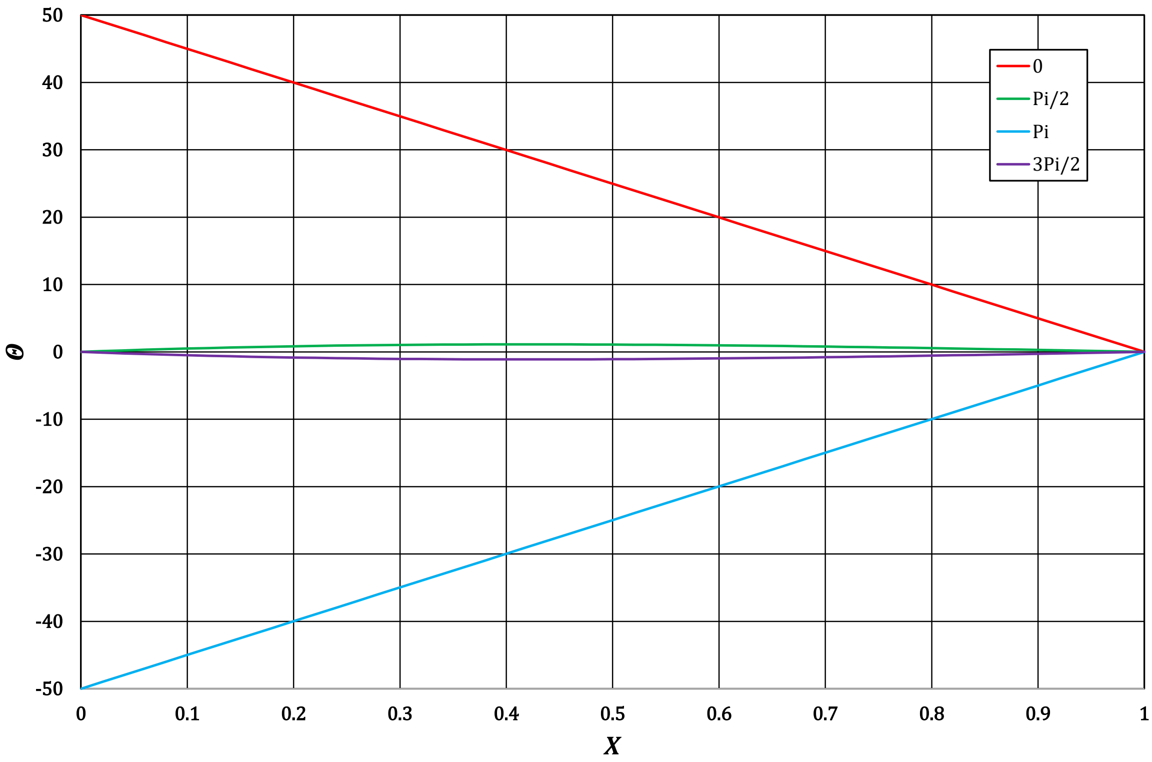
$$\phi(X) = \arctan\left\{\frac{\mathrm{Im}[U(X)]}{\mathrm{Re}[U(X)]}\right\}$$

Columns 6 to 9 show $\Theta(X,T) = A(X)\cos[\Omega T + \phi(X)]$ against X for ΩT values of 0, $\pi/2$, π, and $3\pi/2$. The last four columns are plotted in Figure 3. The diffusivity of the steel is so high that there is little thermal lag in the response of the temperature profiles to the sinusoidal temperature oscillations applied to the outer surface, $X = 0$, of the plate.

Table 1 $\theta(X,T)$ vs. X and ΩT for oscillations on the outer surface of the steel plate

X	$\mathrm{Re}[U(X)]$	$\mathrm{Im}[U(X)]$	$A(X)$	$\phi(X)$	$\phi(X)$	$\theta(X,T)$			
				[Rad]	[Deg]	$\Omega T=0$	$\pi/2$	π	$3\pi/2$
0	50.0000	0.0000	50.0000	0.0000	0.0000	50.0000	0.0000	−50.0000	0.0000
0.02	48.9973	−0.1126	48.9974	−0.0023	−0.1317	48.9973	0.1126	−48.9973	−0.1126
0.04	47.9946	−0.2184	47.9951	−0.0045	−0.2607	47.9946	0.2184	−47.9946	−0.2184
0.06	46.9920	−0.3174	46.9931	−0.0068	−0.3870	46.9920	0.3174	−46.9920	−0.3174
0.08	45.9894	−0.4100	45.9912	−0.0089	−0.5108	45.9894	0.4100	−45.9894	−0.4100
0.10	44.9868	−0.4961	44.9896	−0.0110	−0.6318	44.9868	0.4961	−44.9868	−0.4961
0.12	43.9844	−0.5759	43.9881	−0.0131	−0.7502	43.9844	0.5759	−43.9844	−0.5759
0.14	42.9820	−0.6497	42.9869	−0.0151	−0.8659	42.9820	0.6497	−42.9820	−0.6497
0.16	41.9797	−0.7174	41.9858	−0.0171	−0.9790	41.9797	0.7174	−41.9797	−0.7174
0.18	40.9775	−0.7793	40.9849	−0.0190	−1.0895	40.9775	0.7793	−40.9775	−0.7793
0.20	39.9754	−0.8354	39.9841	−0.0209	−1.1972	39.9754	0.8354	−39.9754	−0.8354
0.22	38.9734	−0.8860	38.9834	−0.0227	−1.3024	38.9734	0.8860	−38.9734	−0.8860
0.24	37.9715	−0.9312	37.9829	−0.0245	−1.4048	37.9715	0.9312	−37.9715	−0.9312
0.26	36.9698	−0.9711	36.9825	−0.0263	−1.5046	36.9698	0.9711	−36.9698	−0.9711
0.28	35.9682	−1.0058	35.9823	−0.0280	−1.6017	35.9682	1.0058	−35.9682	−1.0058
0.30	34.9667	−1.0355	34.9821	−0.0296	−1.6962	34.9667	1.0355	−34.9667	−1.0355
0.32	33.9654	−1.0603	33.9820	−0.0312	−1.7880	33.9654	1.0603	−33.9654	−1.0603
0.34	32.9643	−1.0804	32.9820	−0.0328	−1.8772	32.9643	1.0804	−32.9643	−1.0804
0.36	31.9633	−1.0959	31.9821	−0.0343	−1.9637	31.9633	1.0959	−31.9633	−1.0959
0.38	30.9624	−1.1070	30.9822	−0.0357	−2.0475	30.9624	1.1070	−30.9624	−1.1070
0.40	29.9617	−1.1137	29.9824	−0.0372	−2.1287	29.9617	1.1137	−29.9617	−1.1137
0.42	28.9612	−1.1162	28.9827	−0.0385	−2.2072	28.9612	1.1162	−28.9612	−1.1162
0.44	27.9608	−1.1148	27.9830	−0.0398	−2.2831	27.9608	1.1148	−27.9608	−1.1148
0.46	26.9605	−1.1094	26.9833	−0.0411	−2.3563	26.9605	1.1094	−26.9605	−1.1094
0.48	25.9605	−1.1002	25.9838	−0.0424	−2.4268	25.9605	1.1002	−25.9605	−1.1002
0.50	24.9605	−1.0875	24.9842	−0.0435	−2.4947	24.9605	1.0875	−24.9605	−1.0875
0.52	23.9607	−1.0713	23.9847	−0.0447	−2.5599	23.9607	1.0713	−23.9607	−1.0713
0.54	22.9611	−1.0517	22.9852	−0.0458	−2.6225	22.9611	1.0517	−22.9611	−1.0517
0.56	21.9616	−1.0289	21.9857	−0.0468	−2.6824	21.9616	1.0289	−21.9616	−1.0289
0.58	20.9623	−1.0031	20.9863	−0.0478	−2.7396	20.9623	1.0031	−20.9623	−1.0031
0.60	19.9631	−0.9743	19.9869	−0.0488	−2.7942	19.9631	0.9743	−19.9631	−0.9743
0.62	18.9640	−0.9428	18.9875	−0.0497	−2.8461	18.9640	0.9428	−18.9640	−0.9428
0.64	17.9651	−0.9086	17.9881	−0.0505	−2.8953	17.9651	0.9086	−17.9651	−0.9086
0.66	16.9663	−0.8719	16.9887	−0.0513	−2.9419	16.9663	0.8719	−16.9663	−0.8719
0.68	15.9676	−0.8329	15.9893	−0.0521	−2.9858	15.9676	0.8329	−15.9676	−0.8329
0.70	14.9691	−0.7916	14.9900	−0.0528	−3.0271	14.9691	0.7916	−14.9691	−0.7916
0.72	13.9706	−0.7482	13.9906	−0.0535	−3.0657	13.9706	0.7482	−13.9706	−0.7482
0.74	12.9722	−0.7029	12.9913	−0.0541	−3.1016	12.9722	0.7029	−12.9722	−0.7029
0.76	11.9740	−0.6558	11.9919	−0.0547	−3.1349	11.9740	0.6558	−11.9740	−0.6558
0.78	10.9758	−0.6070	10.9926	−0.0552	−3.1655	10.9758	0.6070	−10.9758	−0.6070
0.80	9.9778	−0.5567	9.9933	−0.0557	−3.1935	9.9778	0.5567	−9.9778	−0.5567
0.82	8.9798	−0.5050	8.9939	−0.0562	−3.2187	8.9798	0.5050	−8.9798	−0.5050
0.84	7.9818	−0.4520	7.9946	−0.0566	−3.2414	7.9818	0.4520	−7.9818	−0.4520
0.86	6.9840	−0.3980	6.9953	−0.0569	−3.2613	6.9840	0.3980	−6.9840	−0.3980
0.88	5.9861	−0.3429	5.9960	−0.0572	−3.2786	5.9861	0.3429	−5.9861	−0.3429
0.90	4.9884	−0.2870	4.9966	−0.0575	−3.2933	4.9884	0.2870	−4.9884	−0.2870
0.92	3.9907	−0.2305	3.9973	−0.0577	−3.3053	3.9907	0.2305	−3.9907	−0.2305
0.94	2.9930	−0.1733	2.9980	−0.0579	−3.3146	2.9930	0.1733	−2.9930	−0.1733
0.96	1.9953	−0.1158	1.9987	−0.0580	−3.3212	1.9953	0.1158	−1.9953	−0.1158
0.98	0.9976	−0.0580	0.9993	−0.0580	−3.3252	0.9976	0.0580	−0.9976	−0.0580
1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Figure 3 $\Theta(X, T)$ vs. X and ΩT for oscillations on the outer surface of the steel plate



(b) The thermal diffusivity α of the timber board is

$$\alpha = \frac{k}{\rho C} = \frac{0.12}{480 \times 1680} = 0.1488 \times 10^{-6}\ \mathrm{m^2\ s^{-1}}$$

so the angular frequency term Ω is

$$\Omega = \frac{2\pi \times 0.02^2}{600 \times 0.1488 \times 10^{-6}} = 28.15\ \mathrm{rad}$$

From the Appendix

$$U(X) = A_0 \frac{(e^a - e^{-a}) \cos a \left[e^{a(1-X)} - e^{-a(1-X)}\right] \cos a(1-X) + (e^a + e^{-a}) \sin a \left[e^{a(1-X)} + e^{-a(1-X)}\right] \sin a(1-X)}{(e^a - e^{-a})^2 \cos^2 a + (e^a + e^{-a})^2 \sin^2 a}$$

$$+ jA_0 \frac{(e^a - e^{-a}) \cos a \left[e^{a(1-X)} + e^{-a(1-X)}\right] \sin a(1-X) - (e^a + e^{-a}) \sin a \left[e^{a(1-X)} - e^{-a(1-X)}\right] \cos a(1-X)}{(e^a - e^{-a})^2 \cos^2 a + (e^a + e^{-a})^2 \sin^2 a} \qquad (\mathrm{A}.1)$$

where

$$a = \sqrt{\frac{\Omega}{2}} = \sqrt{\frac{28.15}{2}} = 3.752$$

Substituting the value of a gives

$$(e^{a} - e^{-a}) \cos a = (e^{3.752} - e^{-3.752}) \cos 3.752 = -34.892903$$

$$(e^{a} + e^{-a}) \sin a = (e^{3.752} + e^{-3.752}) \sin 3.752 = -24.435409$$

$$(e^{a} - e^{-a})^2 \cos^2 a + (e^{a} + e^{-a})^2 \sin^2 a = (-34.892903)^2 + (-24.435409)^2 = 1814.6039$$

$$\frac{(e^{a} - e^{-a}) \cos a}{(e^{a} - e^{-a})^2 \cos^2 a + (e^{a} + e^{-a})^2 \sin^2 a} = \frac{-34.892903}{1814.6039} = -0.019228936$$

$$\frac{(e^{a} + e^{-a}) \sin a}{(e^{a} - e^{-a})^2 \cos^2 a + (e^{a} + e^{-a})^2 \sin^2 a} = \frac{-24.435409}{1814.6039} = -0.013465974$$

Substituting into (A.1) gives

$$U(X) = A_0\left\{(-0.019228936)\left[e^{a(1-X)} - e^{-a(1-X)}\right] \cos a(1-X) + (-0.013465974)\left[e^{a(1-X)} + e^{-a(1-X)}\right] \sin a(1-X)\right\}$$

$$+jA_0\left\{(-0.019228936)\left[e^{a(1-X)} + e^{-a(1-X)}\right] \sin a(1-X) - (-0.013465974)\left[e^{a(1-X)} - e^{-a(1-X)}\right] \cos a(1-X)\right\}$$

where $A_0 = 50°\text{C}$ and $a = 3.752$.

In Table 2, column 1 shows X from 0 to 1 in intervals of 0.02. Columns 2 and 3 show the real and imaginary parts of $U(X)$, column 4 shows the amplitude $A(X)$ of $\Theta(X,T)$:

$$A(X) = \sqrt{\{\mathrm{Re}[U(X)]\}^2 + \{\mathrm{Im}[U(X)]\}^2}$$

and column 5 shows the phase $\phi(X)$ of $\Theta(X,T)$:

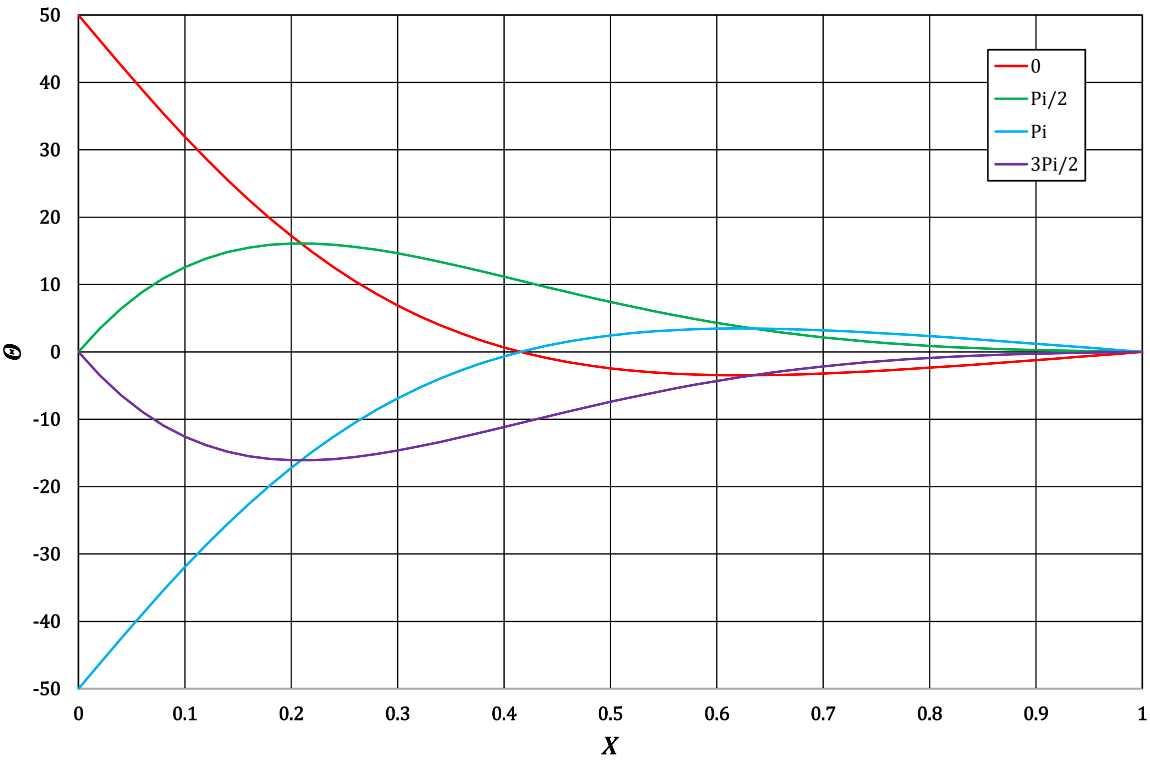
$$\phi(X) = \arctan\left\{\frac{\mathrm{Im}[U(X)]}{\mathrm{Re}[U(X)]}\right\}$$

Columns 6 to 9 show $\Theta(X,T) = A(X)\cos[\Omega T + \phi(X)]$ against X for ΩT values of 0, $\pi/2$, π, and $3\pi/2$. The last four columns are plotted in Figure 4. The diffusivity of the timber is two orders of magnitude smaller than that of steel and consequently there is a much greater time lag in the response of the temperature profiles to the sinusoidal temperature oscillations.

Table 2 $\theta(X,T)$ vs. X and ΩT for oscillations on the outer surface of the timber board

X	$\mathrm{Re}[U(X)]$	$\mathrm{Im}[U(X)]$	$A(X)$	$\phi(X)$	$\phi(X)$	$\theta(X,T)$			
				[Rad]	[Deg]	$\Omega T=0$	$\pi/2$	π	$3\pi/2$
0	50.0000	0.0000	50.0000	0.0000	0.0000	50.0000	0.0000	−50.0000	0.0000
0.02	46.2495	−3.4751	46.3798	−0.0750	−4.2970	46.2495	3.4751	−46.2495	−3.4751
0.04	42.5376	−6.4292	43.0207	−0.1500	−8.5947	42.5376	6.4292	−42.5376	−6.4292
0.06	38.8977	−8.9042	39.9038	−0.2250	−12.8936	38.8977	8.9042	−38.8977	−8.9042
0.08	35.3576	−10.9410	37.0117	−0.3001	−17.1941	35.3576	10.9410	−35.3576	−10.9410
0.10	31.9404	−12.5796	34.3283	−0.3752	−21.4967	31.9404	12.5796	−31.9404	−12.5796
0.12	28.6645	−13.8582	31.8388	−0.4503	−25.8021	28.6645	13.8582	−28.6645	−13.8582
0.14	25.5444	−14.8140	29.5292	−0.5255	−30.1107	25.5444	14.8140	−25.5444	−14.8140
0.16	22.5909	−15.4818	27.3868	−0.6008	−34.4234	22.5909	15.4818	−22.5909	−15.4818
0.18	19.8115	−15.8951	25.3998	−0.6762	−38.7408	19.8115	15.8951	−19.8115	−15.8951
0.20	17.2108	−16.0852	23.5573	−0.7516	−43.0637	17.2108	16.0852	−17.2108	−16.0852
0.22	14.7912	−16.0812	21.8491	−0.8272	−47.3927	14.7912	16.0812	−14.7912	−16.0812
0.24	12.5525	−15.9104	20.2659	−0.9028	−51.7285	12.5525	15.9104	−12.5525	−15.9104
0.26	10.4928	−15.5982	18.7990	−0.9786	−56.0714	10.4928	15.5982	−10.4928	−15.5982
0.28	8.6087	−15.1676	17.4403	−1.0546	−60.4218	8.6087	15.1676	−8.6087	−15.1676
0.30	6.8954	−14.6398	16.1824	−1.1306	−64.7796	6.8954	14.6398	−6.8954	−14.6398
0.32	5.3468	−14.0343	15.0183	−1.2068	−69.1442	5.3468	14.0343	−5.3468	−14.0343
0.34	3.9562	−13.3684	13.9415	−1.2831	−73.5146	3.9562	13.3684	−3.9562	−13.3684
0.36	2.7162	−12.6578	12.9460	−1.3594	−77.8890	2.7162	12.6578	−2.7162	−12.6578
0.38	1.6186	−11.9165	12.0260	−1.4358	−82.2649	1.6186	11.9165	−1.6186	−11.9165
0.40	0.6553	−11.1569	11.1761	−1.5121	−86.6388	0.6553	11.1569	−0.6553	−11.1569
0.42	−0.1824	−10.3897	10.3913	−1.5884	−91.0060	−0.1824	10.3897	0.1824	−10.3897
0.44	−0.9031	−9.6245	9.6668	−1.6644	−95.3608	−0.9031	9.6245	0.9031	−9.6245
0.46	−1.5154	−8.8694	8.9979	−1.7400	−99.6960	−1.5154	8.8694	1.5154	−8.8694
0.48	−2.0278	−8.1312	8.3802	−1.8152	−104.0033	−2.0278	8.1312	2.0278	−8.1312
0.50	−2.4486	−7.4158	7.8096	−1.8897	−108.2728	−2.4486	7.4158	2.4486	−7.4158
0.52	−2.7859	−6.7278	7.2818	−1.9634	−112.4936	−2.7859	6.7278	2.7859	−6.7278
0.54	−3.0473	−6.0712	6.7931	−2.0360	−116.6535	−3.0473	6.0712	3.0473	−6.0712
0.56	−3.2404	−5.4488	6.3396	−2.1073	−120.7396	−3.2404	5.4488	3.2404	−5.4488
0.58	−3.3720	−4.8629	5.9177	−2.1771	−124.7382	−3.3720	4.8629	3.3720	−4.8629
0.60	−3.4489	−4.3149	5.5239	−2.2451	−128.6352	−3.4489	4.3149	3.4489	−4.3149
0.62	−3.4771	−3.8057	5.1550	−2.3111	−132.4166	−3.4771	3.8057	3.4771	−3.8057
0.64	−3.4624	−3.3356	4.8078	−2.3748	−136.0688	−3.4624	3.3356	3.4624	−3.3356
0.66	−3.4102	−2.9045	4.4794	−2.4361	−139.5784	−3.4102	2.9045	3.4102	−2.9045
0.68	−3.3251	−2.5117	4.1672	−2.4947	−142.9335	−3.3251	2.5117	3.3251	−2.5117
0.70	−3.2118	−2.1564	3.8685	−2.5503	−146.1226	−3.2118	2.1564	3.2118	−2.1564
0.72	−3.0741	−1.8372	3.5813	−2.6029	−149.1360	−3.0741	1.8372	3.0741	−1.8372
0.74	−2.9157	−1.5526	3.3033	−2.6523	−151.9648	−2.9157	1.5526	2.9157	−1.5526
0.76	−2.7398	−1.3008	3.0330	−2.6983	−154.6019	−2.7398	1.3008	2.7398	−1.3008
0.78	−2.5492	−1.0799	2.7685	−2.7409	−157.0412	−2.5492	1.0799	2.5492	−1.0799
0.80	−2.3465	−0.8877	2.5088	−2.7799	−159.2777	−2.3465	0.8877	2.3465	−0.8877
0.82	−2.1337	−0.7219	2.2525	−2.8153	−161.3076	−2.1337	0.7219	2.1337	−0.7219
0.84	−1.9127	−0.5801	1.9987	−2.8471	−163.1282	−1.9127	0.5801	1.9127	−0.5801
0.86	−1.6852	−0.4598	1.7468	−2.8752	−164.7372	−1.6852	0.4598	1.6852	−0.4598
0.88	−1.4525	−0.3586	1.4961	−2.8996	−166.1332	−1.4525	0.3586	1.4525	−0.3586
0.90	−1.2158	−0.2736	1.2462	−2.9202	−167.3153	−1.2158	0.2736	1.2158	−0.2736
0.92	−0.9759	−0.2024	0.9967	−2.9371	−168.2829	−0.9759	0.2024	0.9759	−0.2024
0.94	−0.7338	−0.1422	0.7474	−2.9502	−169.0356	−0.7338	0.1422	0.7338	−0.1422
0.96	−0.4900	−0.0902	0.4983	−2.9596	−169.5733	−0.4900	0.0902	0.4900	−0.0902
0.98	−0.2453	−0.0437	0.2491	−2.9652	−169.8959	−0.2453	0.0437	0.2453	−0.0437
1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Figure 4 $\theta(X,T)$ **vs.** X **and** ΩT **for oscillations on the outer surface of the timber board**



6 Example 2

Temperature oscillations are now applied to both sides of the timber board in Example 1. On the outer surface, $x = 0$, the temperature oscillations are the same as those in Example 1. On the inner surface, $x = L$, the temperature oscillates about 0°C with a period of 10 minutes as before, but the amplitude is 25°C and the oscillations lag those on the outer surface by $\pi/2$ rad. The board is initially at 0°C. Calculate the temperature in the board once sufficient time has elapsed for the temperature to become periodic.

To solve this problem, we can make use of the principle of superstition set out in Section 3.3. In Example 1, we solved the dimensionless boundary−value problem

$$\frac{\partial\theta}{\partial T} = \frac{\partial^2\theta}{\partial X^2} \qquad 0 < X < 1, T > 0$$

$$\theta(0, T) = A_0 \cos \Omega T \qquad t \geq 0$$

$$\theta(1, T) = 0 \qquad T \geq 0$$

$$\theta(X, 0) = 0 \qquad 0 < X < 1$$

The values of L and α used to scale the independent variables x, t and ω do not change when temperature oscillations are applied to the inner surface of the board and the solution of the temperature is linear in A_0. Consequently, the temperature solution due to the oscillations on the inner surface of the board is identical to the solution of Example 1, except that the temperature must be scaled by $25/50$ and shifted backwards in time by $\Omega T = \pi/2$. We simply need to modify Table 2 so that X in column 1 runs from 1 to 0, the temperature in columns 7 to 10 is halved, the temperature in the column labelled $\Omega T = 0$ is moved to $\Omega T = \pi/2$, the temperature in the column labelled $\Omega T = \pi/2$ is moved to $\Omega T = \pi$, and so on. These operations have been carried out to produce the temperature solution in Table 3. Temperature profiles at intervals of $\pi/2$ are plotted in Figure 5.

To obtain the solution when temperature oscillations are applied to both sides of the board, we simply need to add together the solutions in Tables 2 and 3. The solution is given in Table 4 and plotted in Figure 6.

Table 3 $\Theta(X,T)$ vs. X and ΩT for oscillations on the inner surface of the timber board

X	$\mathrm{Re}[U(X)]$	$\mathrm{Im}[U(X)]$	$A(X)$	$\phi(X)$	$\phi(X)$	$\Theta(X,T)$			
				[Rad]	[Deg]	$\Omega T = 0$	$\pi/2$	π	$3\pi/2$
1	25.0000	0.0000	25.0000	0.0000	0.0000	0.0000	25.0000	0.0000	−25.0000
0.98	23.1247	−1.7375	23.1899	−0.0750	−4.2970	−1.7375	23.1247	1.7375	−23.1247
0.96	21.2688	−3.2146	21.5104	−0.1500	−8.5947	−3.2146	21.2688	3.2146	−21.2688
0.94	19.4488	−4.4521	19.9519	−0.2250	−12.8936	−4.4521	19.4488	4.4521	−19.4488
0.92	17.6788	−5.4705	18.5059	−0.3001	−17.1941	−5.4705	17.6788	5.4705	−17.6788
0.9	15.9702	−6.2898	17.1642	−0.3752	−21.4967	−6.2898	15.9702	6.2898	−15.9702
0.88	14.3323	−6.9291	15.9194	−0.4503	−25.8021	−6.9291	14.3323	6.9291	−14.3323
0.86	12.7722	−7.4070	14.7646	−0.5255	−30.1107	−7.4070	12.7722	7.4070	−12.7722
0.84	11.2954	−7.7409	13.6934	−0.6008	−34.4234	−7.7409	11.2954	7.7409	−11.2954
0.82	9.9057	−7.9476	12.6999	−0.6762	−38.7408	−7.9476	9.9057	7.9476	−9.9057
0.8	8.6054	−8.0426	11.7786	−0.7516	−43.0637	−8.0426	8.6054	8.0426	−8.6054
0.78	7.3956	−8.0406	10.9245	−0.8272	−47.3927	−8.0406	7.3956	8.0406	−7.3956
0.76	6.2762	−7.9552	10.1329	−0.9028	−51.7285	−7.9552	6.2762	7.9552	−6.2762
0.74	5.2464	−7.7991	9.3995	−0.9786	−56.0714	−7.7991	5.2464	7.7991	−5.2464
0.72	4.3044	−7.5838	8.7202	−1.0546	−60.4218	−7.5838	4.3044	7.5838	−4.3044
0.7	3.4477	−7.3199	8.0912	−1.1306	−64.7796	−7.3199	3.4477	7.3199	−3.4477
0.68	2.6734	−7.0172	7.5092	−1.2068	−69.1442	−7.0172	2.6734	7.0172	−2.6734
0.66	1.9781	−6.6842	6.9708	−1.2831	−73.5146	−6.6842	1.9781	6.6842	−1.9781
0.64	1.3581	−6.3289	6.4730	−1.3594	−77.8890	−6.3289	1.3581	6.3289	−1.3581
0.62	0.8093	−5.9583	6.0130	−1.4358	−82.2649	−5.9583	0.8093	5.9583	−0.8093
0.6	0.3276	−5.5784	5.5881	−1.5121	−86.6388	−5.5784	0.3276	5.5784	−0.3276
0.58	−0.0912	−5.1949	5.1957	−1.5884	−91.0060	−5.1949	−0.0912	5.1949	0.0912
0.56	−0.4516	−4.8122	4.8334	−1.6644	−95.3608	−4.8122	−0.4516	4.8122	0.4516
0.54	−0.7577	−4.4347	4.4989	−1.7400	−99.6960	−4.4347	−0.7577	4.4347	0.7577
0.52	−1.0139	−4.0656	4.1901	−1.8152	−104.0033	−4.0656	−1.0139	4.0656	1.0139
0.5	−1.2243	−3.7079	3.9048	−1.8897	−108.2728	−3.7079	−1.2243	3.7079	1.2243
0.48	−1.3929	−3.3639	3.6409	−1.9634	−112.4936	−3.3639	−1.3929	3.3639	1.3929
0.46	−1.5237	−3.0356	3.3965	−2.0360	−116.6535	−3.0356	−1.5237	3.0356	1.5237
0.44	−1.6202	−2.7244	3.1698	−2.1073	−120.7396	−2.7244	−1.6202	2.7244	1.6202
0.42	−1.6860	−2.4315	2.9588	−2.1771	−124.7382	−2.4315	−1.6860	2.4315	1.6860
0.4	−1.7244	−2.1575	2.7619	−2.2451	−128.6352	−2.1575	−1.7244	2.1575	1.7244
0.38	−1.7386	−1.9029	2.5775	−2.3111	−132.4166	−1.9029	−1.7386	1.9029	1.7386
0.36	−1.7312	−1.6678	2.4039	−2.3748	−136.0688	−1.6678	−1.7312	1.6678	1.7312
0.34	−1.7051	−1.4522	2.2397	−2.4361	−139.5784	−1.4522	−1.7051	1.4522	1.7051
0.32	−1.6626	−1.2559	2.0836	−2.4947	−142.9335	−1.2559	−1.6626	1.2559	1.6626
0.3	−1.6059	−1.0782	1.9343	−2.5503	−146.1226	−1.0782	−1.6059	1.0782	1.6059
0.28	−1.5371	−0.9186	1.7906	−2.6029	−149.1360	−0.9186	−1.5371	0.9186	1.5371
0.26	−1.4579	−0.7763	1.6517	−2.6523	−151.9648	−0.7763	−1.4579	0.7763	1.4579
0.24	−1.3699	−0.6504	1.5165	−2.6983	−154.6019	−0.6504	−1.3699	0.6504	1.3699
0.22	−1.2746	−0.5400	1.3843	−2.7409	−157.0412	−0.5400	−1.2746	0.5400	1.2746
0.2	−1.1732	−0.4438	1.2544	−2.7799	−159.2777	−0.4438	−1.1732	0.4438	1.1732
0.18	−1.0668	−0.3609	1.1262	−2.8153	−161.3076	−0.3609	−1.0668	0.3609	1.0668
0.16	−0.9564	−0.2900	0.9994	−2.8471	−163.1282	−0.2900	−0.9564	0.2900	0.9564
0.14	−0.8426	−0.2299	0.8734	−2.8752	−164.7372	−0.2299	−0.8426	0.2299	0.8426
0.12	−0.7263	−0.1793	0.7481	−2.8996	−166.1332	−0.1793	−0.7263	0.1793	0.7263
0.1	−0.6079	−0.1368	0.6231	−2.9202	−167.3153	−0.1368	−0.6079	0.1368	0.6079
0.08	−0.4880	−0.1012	0.4983	−2.9371	−168.2829	−0.1012	−0.4880	0.1012	0.4880
0.06	−0.3669	−0.0711	0.3737	−2.9502	−169.0356	−0.0711	−0.3669	0.0711	0.3669
0.04	−0.2450	−0.0451	0.2491	−2.9596	−169.5733	−0.0451	−0.2450	0.0451	0.2450
0.02	−0.1226	−0.0219	0.1246	−2.9652	−169.8959	−0.0219	−0.1226	0.0219	0.1226
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Figure 5 $\theta(X,T)$ **vs.** X **and** ΩT **for oscillations on the inner surface of the timber board**

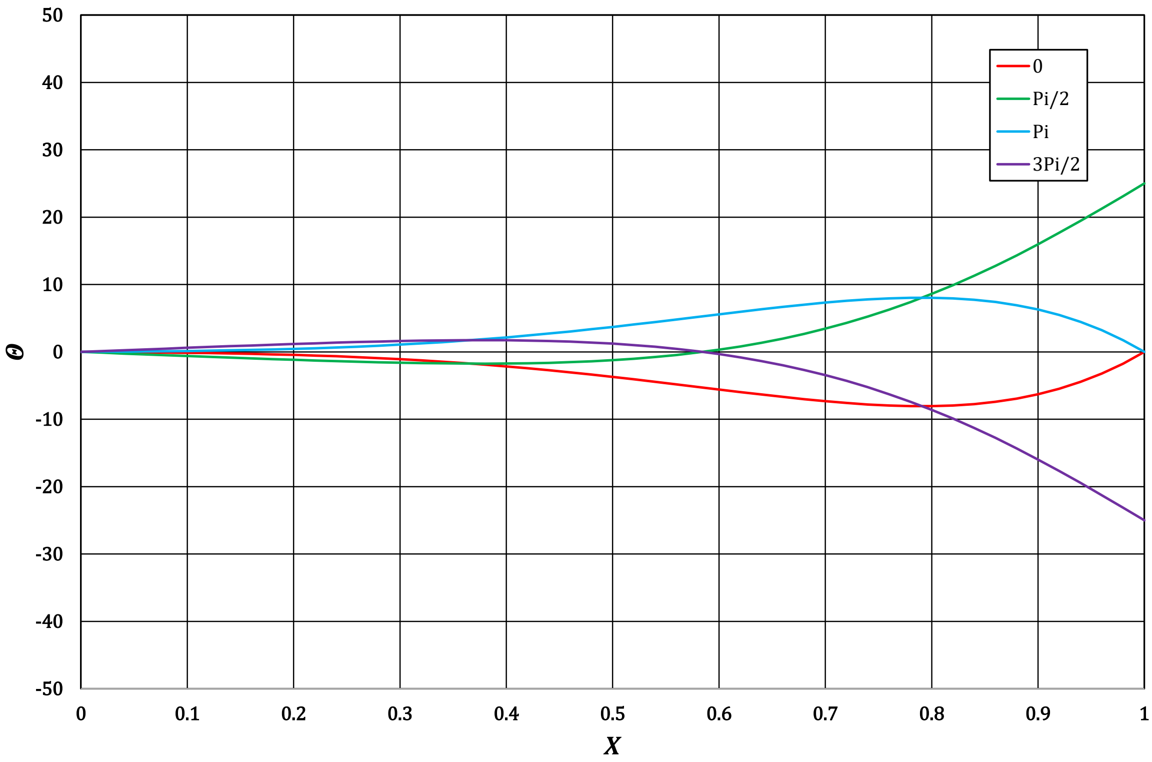
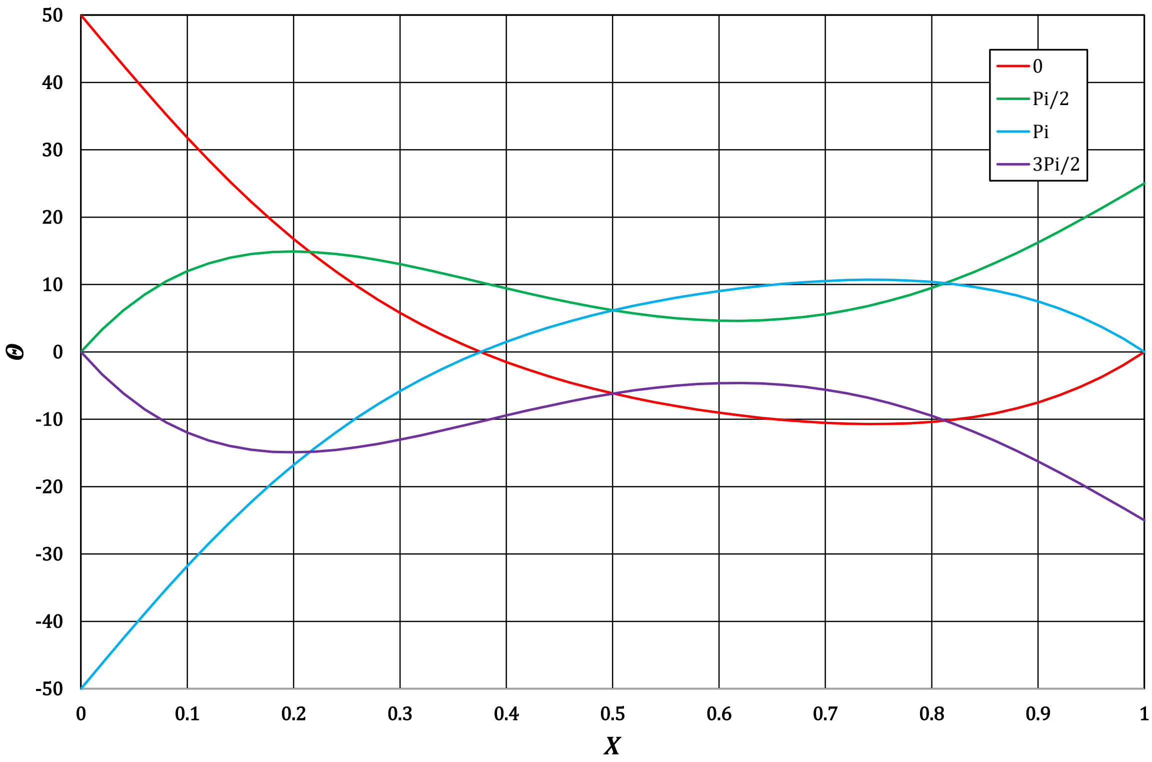


Table 4 $\theta(X,T)$ vs. X and ΩT for oscillations on both sides of the timber board

X	$\theta(X,T)$			
	$\Omega T=0$	$\pi/2$	π	$3\pi/2$
0	50.0000	0.0000	−50.0000	0.0000
0.02	46.2276	3.3524	−46.2276	−3.3524
0.04	42.4925	6.1842	−42.4925	−6.1842
0.06	38.8266	8.5373	−38.8266	−8.5373
0.08	35.2564	10.4531	−35.2564	−10.4531
0.10	31.8036	11.9717	−31.8036	−11.9717
0.12	28.4853	13.1320	−28.4853	−13.1320
0.14	25.3145	13.9714	−25.3145	−13.9714
0.16	22.3008	14.5255	−22.3008	−14.5255
0.18	19.4505	14.8283	−19.4505	−14.8283
0.20	16.7670	14.9119	−16.7670	−14.9119
0.22	14.2512	14.8066	−14.2512	−14.8066
0.24	11.9021	14.5405	−11.9021	−14.5405
0.26	9.7165	14.1403	−9.7165	−14.1403
0.28	7.6901	13.6305	−7.6901	−13.6305
0.30	5.8172	13.0340	−5.8172	−13.0340
0.32	4.0909	12.3718	−4.0909	−12.3718
0.34	2.5040	11.6634	−2.5040	−11.6634
0.36	1.0483	10.9266	−1.0483	−10.9266
0.38	−0.2842	10.1780	0.2842	−10.1780
0.40	−1.5022	9.4324	1.5022	−9.4324
0.42	−2.6139	8.7037	2.6139	−8.7037
0.44	−3.6276	8.0043	3.6276	−8.0043
0.46	−4.5510	7.3457	4.5510	−7.3457
0.48	−5.3917	6.7382	5.3917	−6.7382
0.50	−6.1565	6.1914	6.1565	−6.1914
0.52	−6.8515	5.7139	6.8515	−5.7139
0.54	−7.4820	5.3135	7.4820	−5.3135
0.56	−8.0526	4.9973	8.0526	−4.9973
0.58	−8.5669	4.7717	8.5669	−4.7717
0.60	−9.0273	4.6425	9.0273	−4.6425
0.62	−9.4354	4.6150	9.4354	−4.6150
0.64	−9.7914	4.6937	9.7914	−4.6937
0.66	−10.0944	4.8826	10.0944	−4.8826
0.68	−10.3423	5.1851	10.3423	−5.1851
0.70	−10.5317	5.6041	10.5317	−5.6041
0.72	−10.6579	6.1416	10.6579	−6.1416
0.74	−10.7148	6.7990	10.7148	−6.7990
0.76	−10.6950	7.5771	10.6950	−7.5771
0.78	−10.5898	8.4755	10.5898	−8.4755
0.80	−10.3890	9.4931	10.3890	−9.4931
0.82	−10.0812	10.6276	10.0812	−10.6276
0.84	−9.6536	11.8755	9.6536	−11.8755
0.86	−9.0922	13.2321	9.0922	−13.2321
0.88	−8.3816	14.6908	8.3816	−14.6908
0.90	−7.5055	16.2438	7.5055	−16.2438
0.92	−6.4464	17.8812	6.4464	−17.8812
0.94	−5.1859	19.5910	5.1859	−19.5910
0.96	−3.7046	21.3590	3.7046	−21.3590
0.98	−1.9828	23.1684	1.9828	−23.1684
1.00	0.0000	25.0000	0.0000	−25.0000

Figure 6 $\Theta(X, T)$ vs. X and ΩT for oscillations on both sides of the timber board



7 Example 3

This example is the same as Example 2, except that the temperature on the outer surface of the board oscillates about 40°C and the temperature on the inner surface oscillates about 10°C.

To calculate the temperature in the board, we can use the principle of superposition again. We need to solve two more boundary-value problems:

$$\frac{\partial \theta}{\partial T} = \frac{\partial^2 \theta}{\partial X^2} \qquad 0 < X < 1, T > 0$$

$$\theta(0, T) = 40 \qquad T \geq 0$$

$$\theta(1, T) = 0 \qquad T \geq 0$$

$$\theta(X, 0) = 0 \qquad 0 < X < 1$$

and

$$\frac{\partial \theta}{\partial T} = \frac{\partial^2 \theta}{\partial X^2} \qquad 0 < X < 1, T > 0$$

$$\theta(0, T) = 0 \qquad T \geq 0$$

$$\theta(1, T) = 10 \qquad T \geq 0$$

$$\theta(X, 0) = 0 \qquad 0 < X < 1$$

The dimensionless temperature θ in the slab is initially zero everywhere. In both problems, the temperatures on both sides of the board are fixed, so after a period of time has elapsed the temperature profile across the board will become steady. The two problems reduce to

$$\frac{d^2 \theta}{dX^2} = 0 \qquad 0 < X < 1$$

$$\theta = 40 \qquad X = 0$$

$$\theta = 0 \qquad X = 1$$

and

$$\frac{d^2 \theta}{dX^2} = 0 \qquad 0 < X < 1$$

$$\theta = 0 \qquad X = 0$$

$$\theta = 10 \qquad X = 1$$

The solution to the first problem is simply a straight line between $X = 0$, $\theta = 40$ and $X = 1$, $\theta = 0$ and the solution to the second problem is simply a straight line between $X = 0$, $\theta = 0$ and $X = 1$, $\theta = 10$, as shown in Figure 7.

To obtain the temperature solution in this example we add together the two straight lines in Figure 7. The resulting straight line is the time-averaged temperature in the example. Then we add this straight line to each of the profiles in Figure 6. This has been done in Figure 8.

Figure 7 Solutions to the third and fourth boundary-value problems

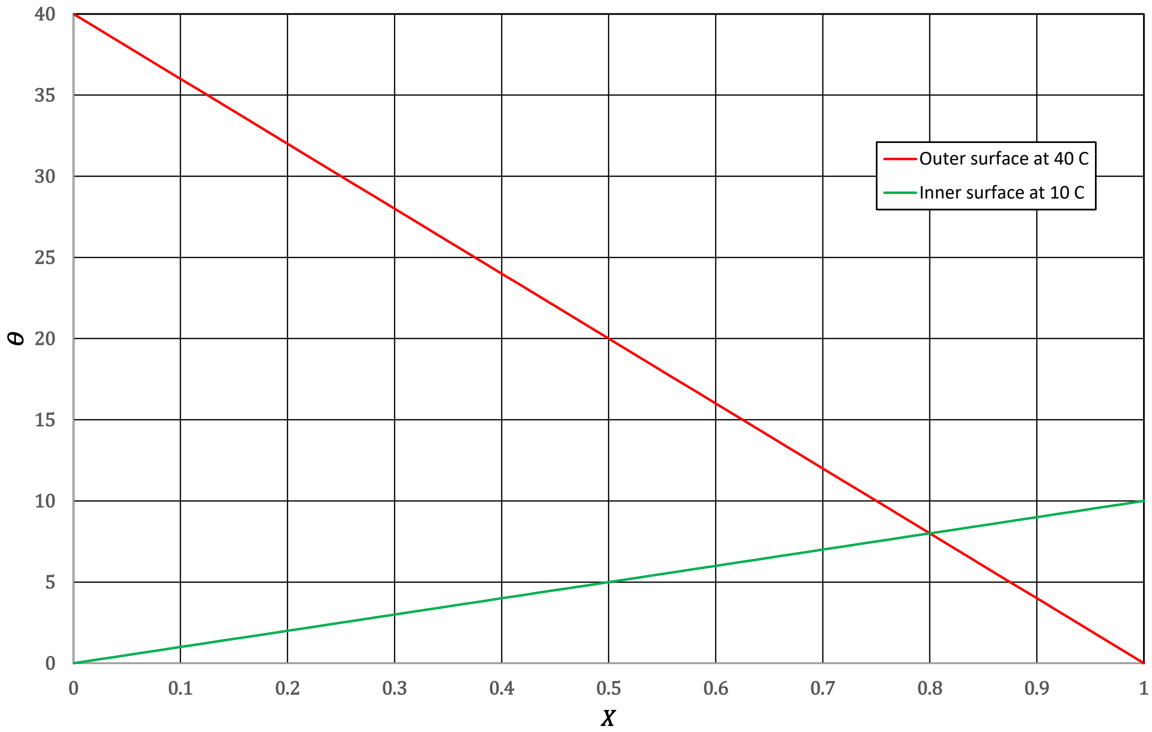
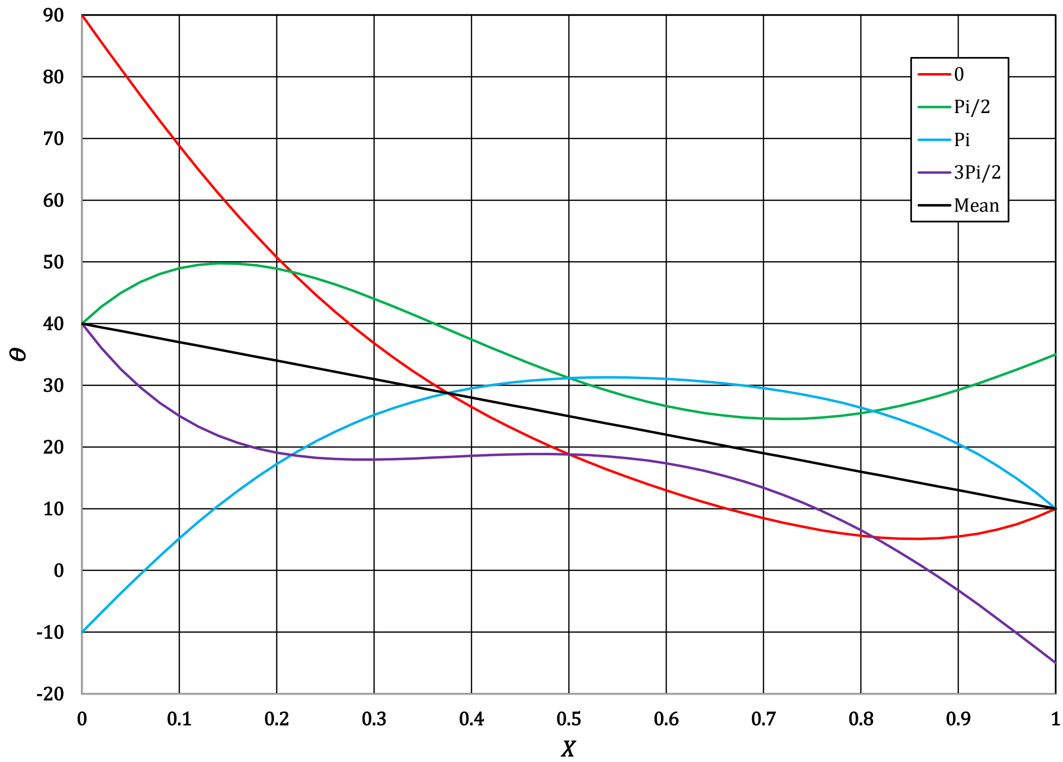


Figure 8 $\theta(X, T)$ vs. X and ΩT with straight-line mean value



8 References

1. K. N. Atkinson, *Admittance Method. 2. Mathematical Development. Theory Guide*, Atkinson Science Limited, 2020.*
2. K. N. Atkinson, *Admittance Method. 3. Composite Wall. Theory Guide*, Atkinson Science Limited, 2020.*
3. K. N. Atkinson, *Admittance Method. 4. Convection and Radiation. Theory Guide*, Atkinson Science Limited, 2020.*
4. K. N. Atkinson, *Admittance Method. 5. Dynamic Thermal Parameters. Theory Guide*, Atkinson Science Limited, 2020.*
5. *CIBSE Guide A, Environmental design*, Chartered Institution of Building Services Engineers, 2015.
6. M. R. Spiegel, *Schaum's Outline of Advanced Mathematics for Engineers and Scientists*, McGraw-Hill, 1971.
7. R. B. Guenther and J. W. Lee, *Partial Differential Equations for Mathematical Physics and Integral Equations*, Dover Publications Inc. 24 June 1996.

* Download from https://atkinsonscience.co.uk/Downloads/Construction.aspx

9 Appendix

In this Appendix we show how the complex function $U(X)$ given by (5.2) can be split into a real part and an imaginary part. Recall (5.2):

$$U(X) = A_0 \frac{\exp\left[\sqrt{\frac{\Omega}{2}}(1+j)(1-X)\right] - \exp\left[-\sqrt{\frac{\Omega}{2}}(1+j)(1-X)\right]}{\exp\left[\sqrt{\frac{\Omega}{2}}(1+j)\right] - \exp\left[-\sqrt{\frac{\Omega}{2}}(1+j)\right]}$$

Let $a = \sqrt{\Omega/2}$, then

$$U(X) = A_0 \frac{\exp[a(1+j)(1-X)] - \exp[-a(1+j)(1-X)]}{\exp[a(1+j)] - \exp[-a(1+j)]}$$

$$= A_0 \frac{e^{a(1-X)}e^{ja(1-X)} - e^{-a(1-X)}e^{-ja(1-x)}}{e^a e^{ja} - e^{-a}e^{-ja}}$$

$$= A_0 \frac{e^{a(1-X)}[\cos a(1-X) + j\sin a(1-X)] - e^{-a(1-X)}[\cos a(1-X) - j\sin a(1-X)]}{e^a(\cos a + j\sin a) - e^{-a}(\cos a - j\sin a)}$$

$$= A_0 \frac{\left[e^{a(1-X)} - e^{-a(1-X)}\right]\cos a(1-X) + j\left[e^{a(1-X)} + e^{-a(1-X)}\right]\sin a(1-X)}{(e^a - e^{-a})\cos a + j\,(e^a + e^{-a})\sin a}$$

Multiplying top and bottom by $(e^a - e^{-a}) \cos a - j\,(e^a + e^{-a})\sin a$:

$$U(X) = A_0 \frac{(e^a - e^{-a}) \cos a \left[e^{a(1-X)} - e^{-a(1-X)}\right] \cos a(1-X) + j(e^a - e^{-a}) \cos a \left[e^{a(1-X)} + e^{-a(1-X)}\right] \sin a(1-X)}{(e^a - e^{-a})^2 \cos^2 a + (e^a + e^{-a})^2 \sin^2 a}$$

$$+A_0 \frac{-j(e^a + e^{-a}) \sin a \left[e^{a(1-X)} - e^{-a(1-X)}\right] \cos a(1-X) + (e^a + e^{-a}) \sin a \left[e^{a(1-X)} + e^{-a(1-X)}\right] \sin a(1-X)}{(e^a - e^{-a})^2 \cos^2 a + (e^a + e^{-a})^2 \sin^2 a}$$

so

$$U(X) = A_0 \frac{(e^a - e^{-a}) \cos a \left[e^{a(1-X)} - e^{-a(1-X)}\right] \cos a(1-X) + (e^a + e^{-a}) \sin a \left[e^{a(1-X)} + e^{-a(1-X)}\right] \sin a(1-X)}{(e^a - e^{-a})^2 \cos^2 a + (e^a + e^{-a})^2 \sin^2 a}$$

$$+jA_0 \frac{(e^a - e^{-a}) \cos a \left[e^{a(1-X)} + e^{-a(1-X)}\right] \sin a(1-X) - (e^a + e^{-a}) \sin a \left[e^{a(1-X)} - e^{-a(1-X)}\right] \cos a(1-X)}{(e^a - e^{-a})^2 \cos^2 a + (e^a + e^{-a})^2 \sin^2 a} \quad \text{(A. 1)}$$

When $X = 0$,

$$U(X) = A_0 \frac{(e^a - e^{-a})^2 \cos^2 a + (e^a + e^{-a})^2 \sin^2 a}{(e^a - e^{-a})^2 \cos^2 a + (e^a + e^{-a})^2 \sin^2 a}$$

$$+jA_0 \frac{(e^a - e^{-a})(e^a + e^{-a}) \cos a \sin a - (e^a + e^{-a})(e^a - e^{-a}) \sin a \cos a}{(e^a - e^{-a})^2 \cos^2 a + (e^a + e^{-a})^2 \sin^2 a}$$

$$= A_0 + j0$$

The magnitude $A(X)$ of $U(X)$ at $X = 0$ is

$$A(X) = |U(X)| = \sqrt{\{\mathrm{Re}[U(X)]\}^2 + \{\mathrm{Im}[U(X)]\}^2} = \sqrt{A_0^2 + 0^2} = A_0$$

and the phase $\phi(X)$ of $U(X)$ at $X = 0$ is

$$\phi(X) = \arctan\left\{\frac{\mathrm{Im}[U(X)]}{\mathrm{Re}[U(X)]}\right\} = \arctan\left\{\frac{0}{A_0}\right\} = 0$$

so

$$\Theta(X,T) = A(X)\cos[\Omega T + \phi(X)]$$

$$= A_0 \cos \Omega T$$

as expected.

When $X = 1$,

$$U(X) = A_0 \frac{(e^a - e^{-a})\cos a\,[e^{a0} - e^{-a0}]\cos a0 + (e^a + e^{-a})\sin a\,[e^{a0} + e^{-a0}]\sin a0}{(e^a - e^{-a})^2\cos^2 a + (e^a + e^{-a})^2\sin^2 a}$$

$$+jA_0 \frac{(e^a - e^{-a})\cos a\,[e^{a0} + e^{-a0}]\sin a0 - (e^a + e^{-a})\sin a\,[e^{a0} - e^{-a0}]\cos a0}{(e^a - e^{-a})^2\cos^2 a + (e^a + e^{-a})^2\sin^2 a}$$

$$= 0 + j0$$

so

$$\Theta(X,T) = 0$$

as expected.